APPLICATION OF THE LAPLACE–CARSON INTEGRAL TRANSFORM METHOD TO THE THEORY OF NONSTATIONARY FLOWS OF A VISCOPLASTIC MEDIUM

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We show that with the application of aLaplace–Carson integral transform the problem concerning the development of a gradient flow of a viscoplastic medium between parallel walls reduces to the solution of a system of functional equations.

It is a known fact that the study of the nonstationary flow of an imcompressible viscoplastic medium in a two-dimensional channel ($-h \le y \le h$) under the influence of a pressure gradient may be reduced to the solution of the following problem [1]:

$$\tau = \mu \frac{\partial v_x}{\partial y} + \tau_0 \operatorname{sign} \frac{\partial v_x}{\partial y}, \quad |\tau| > \tau_0; \tag{1}$$

$$\frac{\partial v_x}{\partial y} = 0, \quad |\tau| \leqslant \tau_0; \tag{2}$$

$$\rho \frac{\partial v_x}{\partial t} = -\frac{\partial p}{\partial x} + \frac{\partial \tau}{\partial y}; \tag{3}$$

$$\rho = \text{const}, \; v_y = v_z = 0, \quad \frac{\partial v_x}{\partial x} = \frac{\partial v_x}{\partial z} = 0, \quad \frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0; \tag{4}$$

$$\begin{gathered} v_x(h, t) = v_x(-h, t) = 0; \\ \tau\left(\pm y_0(t), t\right) = \mp \tau_0, \quad \frac{\partial v_x}{\partial y}\left(\pm y_0(t), t\right) = 0; \end{gathered} \tag{5}$$

$$v_x(y, 0) = \begin{cases} F(y) & \text{for } y_0(0) < y < h, \; -h < y < y_0(0), \\ F\left(y_0(0)\right) & \text{for } -y_0(0) < y \leqslant y_0(0), \end{cases} \tag{6}$$

where the abscissa is taken along the flow direction, the ordinate is perpendicular to the channel walls, and the z-axis is perpendicular to the flow direction; Eqs. (1) describe the rheological behavior of a viscoplastic medium, Eqs. (2) and (3) describe the motion of a continuous medium between parallel walls; conditions (4) express the "no slip" condition of the medium at the nonmoving rigid walls; relations (5) are the conditions for the existence of a zone of quasi-rigid motion ("kernel"), the coordinate of whose boundary is describable by the equation $y = y_0(t)$. Equation (6) gives the initial velocity distribution of the medium.

From Eqs. (2) and (3) it follows that $\partial p/\partial x = \partial p/\partial x(t)$. In this case, taking into account Eqs. (5) for the velocity of the quasi-rigid zone, we readily obtain

$$\rho \frac{dv_0}{dt} = -\frac{\partial p}{\partial x} - \frac{\tau_0}{y_0(t)}. \tag{7}$$

A natural requirement in this problem is the condition of continuity of the velocity of the medium and of the tangential shear stresses at the boundary separating the zones of viscous flow and quasi-rigid motion:

$$v_x\left(\pm y_0(t) \pm 0, t\right) = v_0(t), \quad \tau\left(\pm y_0(t) \mp 0, t\right) = \mp \tau_0. \tag{8}$$

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Owing to the symmetry of the problem defined by Eqs. (1)-(8) it is sufficient to obtain its solution in the domain $\{y, t: y_0(t) < y < h, t > 0\}$. Upon introducing the dimensionless quantities: time $\xi = (\mu/\rho h^2)t$, the coordinate $\eta = y/h$, the velocity $u = (\mu l/p_0 h^2)v_x$, and the tangential stress $T = l/p_0 h[\tau + \tau_0]$, we can write the problem (1)-(8) in the two equivalent forms:

u- representation $\qquad\qquad$ τ- representation

$$\frac{\partial u}{\partial \xi} = \frac{\partial^2 u}{\partial \eta^2} - \hat{p}(\xi), \qquad \frac{\partial T}{\partial \xi} = \frac{\partial^2 T}{\partial \eta^2}, \tag{9}$$

$$u(1, \xi) = 0, \qquad \frac{\partial T}{\partial \eta}(1, \xi) = \hat{p}(\xi), \tag{10}$$

$$\frac{\partial u}{\partial \eta}(\delta(\xi), \xi) = 0, \qquad T(\delta(\xi), \xi) = 0, \tag{11}$$

$$u(\delta(\xi), \xi) = u_0(\xi), \qquad \frac{\partial T}{\partial \eta}(\delta(\xi), \xi) = \frac{\partial T_0}{\partial \eta}, \tag{12}$$

$$u_0(\xi) = u_0(0) - \int_0^{\xi} \left[\hat{p}(\sigma) + \frac{s}{\delta(\sigma)}\right] d\sigma, \qquad \frac{\partial T_0}{\partial \eta} = -\frac{s}{\delta(\xi)}, \tag{13}$$

$$u(\eta, 0) = \hat{F}(\eta) = \begin{cases} \dfrac{\mu l}{p_0 h^2} F(y) \\ \dfrac{\mu l}{p_0 h^2} F(y_0(0)) \end{cases},$$

$$T(\eta, 0) = \hat{\Phi}(\eta) = \begin{cases} \dfrac{d\hat{F}}{d\eta} & \text{for } \delta(0) < \eta < 1 \\ s\left[1 - \dfrac{\eta}{\delta(0)}\right] & \text{for } 0 \leqslant \eta \leqslant \delta(0) \end{cases}, \tag{14}$$

where

$$\hat{p}(\xi) = \frac{l}{p_0}\frac{\partial p}{\partial x}; \quad \delta(\xi) = \frac{y_0(t)}{h}; \quad u_0 = \frac{\mu l}{p_0 h^2} v_0;$$

T_0 is the dimensionless tangential stress distribution in the "main body" of the flow. In the problem defined by Eqs. (9)-(14) the dimensionless parameter $s = \tau_0 l/p_0 h$ characterizes the viscoplasticity of the body; when $s = 0$ the flow becomes that of a Newtonian liquid.

It should be noted that the Eqs. (9) are valid only in the region of viscous flow.

The problem defined by Eqs. (9)-(14) in the u-representation was considered for the first time, apparently, in [1], wherein existence and uniqueness theorems concerning the solution were proved and an asymptotic solution of the problem was obtained for small ξ for $\hat{p}(\xi) = -1$, $u_0(0) = 0$, $\delta(0) = 1$. In the general case, the determination of the boundary $\delta(\xi)$ separating the zones was reduced to the solution of two independent functional equations. A similar problem for more general conditions on the boundary of the main body of the flow was considered in [2]. The results given in [1] and in [2] for identical initial and boundary conditions (including the conditions on the unknown boundary) are coincident.

A survey is given in [3], which bears on the study of nonstationary flows of viscoplastic media, in which the current state of the problem is discussed. In [4], to solve the problem concerning the development of the motion of a viscoplastic medium from a state of rest under the action of an instantaneously applied pressure gradient, constant in time, in a two-dimensional channel, i.e., in solving the problem for the case

$$\hat{p}(\xi) = -1, \quad \delta(0) = 1, \tag{15}$$

the one-sided Laplace transform with respect to the variable ξ was used. However in [4], in the formulation of the problem in the τ-representation, errors were made which lead to invalid results in the final calculation. In the present paper we correct these errors and show that the application of the Laplace–Carson integral transform to the problem defined by Eqs. (9)-(13) and (15) reduces it to the solution of a system of functional equations.

It should be noted that if the conditions (15) hold, the influence of the initial conditions (14) on the solution of problem (9)-(15) is unimportant in the closed region $\bar{D} = \{\eta, \xi : \delta(\xi) \leq \eta \leq 1, 0 \leq \xi \leq \bar{\xi}\}$, where $\bar{\xi}$ is a positive constant and the point (1; 0) is excluded. The solution of the problem (9)-(13), (15) can be constructed by using the method of extending the initial conditions [5], the essence of which consists in the fact that in some region, containing the region D, a thermal source density distribution can be chosen so that the boundary conditions of the problem are satisfied. This method is equivalent, in a certain sense, to the assignment of arbitrary boundary (or initial) conditions for constructing the solution of the heat conduction equation in the domain D, or some extension of it, such that the resulting solution will satisfy the conditions of the initial problem (9)-(13), (15). We examine below two possible ways of constructing the solution to this problem.

We may seek the solution of the problem (9)-(13), (15) in the class of solutions of the heat conduction equation for the halfstrip $\{0 < \eta < \infty,\ 0 < \xi < \bar{\xi}\}$ with an arbitrary initial condition $\varphi(\eta)$. To be specific, we consider the τ-representation of the problem and apply to the Eqs. (9) and (10) the one-sided Laplace–Carson transform, with parameter p, in the variable ξ; we readily obtain

$$p\bar{T} - p\varphi(\eta) = \bar{T}'', \tag{16}$$

$$\bar{T}'(1, p) = -1, \tag{17}$$

where the primes indicate differentiation with respect to the variable η. A solution of Eq. (16), which satisfies the condition (17) and remains bounded as $\eta \to \infty$, has the form

$$\bar{T}(\eta, p) = \frac{\exp\left(\sqrt{p}(1-\eta)\right)}{\sqrt{p}} - e^{-\sqrt{p}\eta} \int_1^\eta e^{2\sqrt{p}z}\, dz \int_1^z e^{-\sqrt{p}r} p\varphi(r)\, dr. \tag{18}$$

However the expression (18) does not satisfy the conditions imposed on the function represented by the Laplace–Carson integral [7].

The second way of solving the problem (9)-(13), (15) consists in arbitrarily assigning a boundary condition on some line $\eta = a$, where a is a bounded constant. Thus the solution is constructed in the region $\{a < \eta < 1,\ 0 < \xi < \bar{\xi}\}$, if $a < 1$; the initial conditions, by virtue of the remarks made above, can be taken to be homogeneous. On the line $\eta = 0$, let there by given the arbitrary boundary condition

$$T(0, \xi) = f(\xi). \tag{19}$$

By taking the Laplace–Carson transform in the variable ξ of the Eqs. (9), (10), with Eqs. (15) and (19) taken into account, we obtain

$$\bar{T}'' - p\bar{T} = 0; \quad \bar{T}(0, p) = \bar{f}(p), \quad \bar{T}'(1, p) = -1. \tag{20}$$

The solution of problem (20) has the form

$$\bar{T}(\eta, p) = \bar{f}(p)\, \frac{\operatorname{ch}\sqrt{p}(1-\eta)}{\operatorname{ch}\sqrt{p}} - \frac{\operatorname{sh}\sqrt{p}\,\eta}{\sqrt{p}\operatorname{ch}\sqrt{p}}. \tag{21}$$

Taking the inverse transform [6], we readily obtain

$$T(\eta, \xi) = -\int_0^\xi \frac{\partial}{\partial\eta} \vartheta_2\left(\frac{\eta}{2},\ \xi-\sigma\right) f(\sigma)\, d\sigma$$

$$-\int_0^\xi \vartheta_1\left(\frac{\eta}{2},\ \sigma\right) d\sigma, \tag{22}$$

where

$$\vartheta_1(y, x) = 2\sum_{k=0}^{\infty} \exp\left[-\pi^2\left(k+\frac{1}{2}\right)^2 x\right] \sin\pi(2k+1)\, y;$$

$$\vartheta_2(y, x) = 2\sum_{k=0}^{\infty} \exp\left[-\pi^2\left(k+\frac{1}{2}\right)^2 x\right] \cos\pi(2k+1)\, y.$$

The system of functional equations for determining $\delta(\xi)$ and $f(\xi)$ has the form

$$\int_0^\xi \frac{\partial}{\partial\eta}\vartheta_2\left(\frac{\delta(\xi)}{2},\ \xi-\sigma\right)f(\sigma)\,d\sigma+\int_0^\xi \vartheta_1\left(\frac{\delta(\xi)}{2},\ \sigma\right)d\sigma=0,$$

$$\int_0^\xi \frac{\partial^2}{\partial\eta^2}\vartheta_2\left(\frac{\delta(\xi)}{2},\ \xi-\sigma\right)f(\sigma)\,d\sigma$$

$$+\int_0^\xi \frac{\partial\vartheta_1}{\partial\eta}\left(\frac{\delta(\xi)}{2},\ \sigma\right)d\sigma=\frac{s}{\delta(\xi)}.$$

We can try to obtain analogous solutions and functional systems for an arbitrarily given boundary condition of the form

$$T(1,\ \xi)=f_1(\xi)\quad\text{or}\quad \frac{\partial T}{\partial\eta}(0,\ \xi)=f_2(\xi).$$

It should be noted that the Laplace–Carson integral transform, owing to its linearity, is not available for handling the conditions on the unknown boundary, whence it follows that to determine the latter it becomes necessary to solve a functional equation.

The error in [4] consists in the fact that in using the second of the schemes considered above satisfaction of the condition $\tau(0, t) = 0$ was required instead of the condition (19). This condition, in actuality, must be satisfied for the zone of quasi-rigid motion and has no relationship to the zone of viscous flow since the line $\eta = 0$ is not a boundary of the viscous flow zone. It would seem that one could arrive at the results given in [4] by taking into account the nature of the tangential shear stress distribution in the two-dimensional channel, keeping only an antisymmetric function of η in the solution. However even this argument falls short since the nature of the tangential stress distribution in the viscous zone of the lower half of the channel is ensured with a suitable choice of the constants of integration.

In addition, in constructing the solution in [4] no use was made of Eqs. (12) and (13), this being a consequence of considering the motion of the quasi-rigid "core" as the entire motion. The true asymptotic nature of the behavior of $\delta(\xi)$ as $\xi\to\infty$ may be explained by the fact that we have considered a flow asymptotically tending to a stationary flow, and for a stationary flow of a viscoplastic medium between parallel walls the equations for the tangential stresses in the zone of viscous flow and in the zone of quasi-rigid motion coincide.

NOTATION

h	half-width of channel;
τ	shear stress;
μ	dynamic viscosity coefficient;
τ_0	limit shear stress;
v_x, v_y, v_z	velocity components of medium along corresponding coordinate axes;
ρ	density of medium;
p	pressure;
v_0	velocity of quasi-rigid zone;
t	time;
p_0/l	pressure drop per unit length;
ξ	dimensionless time;
η	dimensionless transverse coordinate;
δ	dimensionless coordinate of quasi-rigid zone;
u	dimensionless velocity of medium;
T	dimensionless shear stress;
s	plasticity parameter;
$\hat{p}$	dimensionless pressure gradient.

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